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STOCHASTIC SYSTEMS

Problems in the Theory of H^2/H_∞ Controllers for Linear Stochastic Multiplicative-Type Plants

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Abstract—This paper considers H^2/H_∞ control problems for dynamic plants described by linear Itô stochastic equations with the drift and diffusion coefficients linearly dependent on the state vector, control input, and an exogenous disturbance. The controlled plant has two outputs, namely, the regulated z and the observed (noisy) y ones. The controller is optimized by the quadratic H^2 criterion under the boundedness condition for the induced norm of the operator $H_{z\upsilon}$ relating the exogenous disturbance υ to the regulated output $z : \|H_{z\upsilon}\|_\infty < \gamma$. The conditional H^2/H_∞ optimization problem is solved using differential game theory.

Keywords: H^2/H_∞ control theory, Itô diffusion equation, multiplicative stochastic system, induced operator norm, regulated output, output-feedback controller

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1. INTRODUCTION

In this paper, we present some results on the theory of H^2/H_∞ controllers for time-varying plants described by Itô stochastic equations whose drift and diffusion coefficients depend linearly on the state x, control u, and exogenous disturbance υ vectors. These are Itô equations of the form

$$dx(t) = \varphi(t)dt + \Phi(t)dW(t), \quad \varphi = Ax + B_1u + B_2\upsilon, \tag{1.1}$$
$$\Phi\, dW = A_0xdw_0 + B_{01}udw_1 + B_{02}\upsilon dw_2.$$

Such a specific structure of the drift and diffusion coefficients gives a reason to call equation (1.1) *multiplicative* in each of the three variables mentioned. By assumption, the stochastic processes $w_i(t)$, $i = 0, 1, 2$, are scalar, which is actually not a restriction. The dependence of the drift $\varphi(t)$ and diffusion $\Phi(t)dW(t)$ on the same disturbance $\upsilon(t)$ is not a restriction as well. Finally, the processes $w_i(t)$ are supposed to be statistically dependent, not necessarily with a unit intensity matrix.

The plant has two outputs, i.e., the regulated z and observed y ones:

$$\begin{cases} z(t) = C_1x(t) + D_{11}u(t) + D_{12}\upsilon(t) \\ y(t) = C_2x(t) + D_{21}u(t) + D_{22}\upsilon(t), \quad t \in [0, T]. \end{cases} \tag{1.2}$$

In the nonstationary case, the matrix coefficients depend on the time parameter t. We consider the cases of finite ($T < \infty$) and infinite ($T = \infty$) horizons $[0, T]$. Let x_0 denote the initial condition of equation (1.1).

In control theory, the presence of regulated and observed outputs is natural for practical problem statements. Already at the initial stage of optimization theory, it was necessary to distinguish between *LQR* problems (the design of linear *quadratic* regulators based on the full output z) and *LQG* problems (the design of linear quadratic *Gaussian* regulators based on the information contained in the partially observed output y). However, a well-known fact is that in the latter case with disturbances, it is not always possible to ensure the required *robustness* of the regulator. Generally speaking, the robustness property is one of the fundamental requirements for controller design in the modern H^2/H_∞ control theory and especially in the theory of the so-called *uncertain* systems. Note that in the presence of exogenous disturbances and endogenous perturbations, inherent to an uncertain plant, robustness is the most important issue, not only in controller design but also in *filtering*. However, filtering problems are not considered below, like control optimization and filtering problems for *discrete systems*. Here, it seems appropriate to draw the reader's attention to the similarity of many results of the deterministic control theory with some results of the stochastic H^2/H_∞ theory, especially those concerning plants with a state-feedback controller in Section 2. The main topic of this paper is the stochastic theory of robust control of systems (1.1)–(1.2).

The design of an H^2/H_∞ controller is a conditional (constrained) quadratic optimization problem under an upper bound $\gamma > 0$ imposed on the operator norm $\|H_{zv}\|_\infty$ of the operator H_{zv} with the domain and codomain defined by the functional spaces of all plant's input disturbances $v(\cdot)$ and regulated outputs $z(\cdot)$, respectively. The controller must stabilize the closed-loop system and ensure the boundedness condition $\|H_{zv}\|_\infty < \gamma$ for the induced norm. In the general H^2/H_∞ control theory, the description of such a class of controllers constitutes a subbranch designated by the theory of H_∞ controllers.

In the theory of time-invariant systems, it is conventional to consider a class of controllers of the form $u(t) = \mathcal{K}(t, x(\cdot)|_0^t)$, $t > 0$, where the function $\mathcal{K}$ is Borel measurable and Lipschitz continuous in the second argument. If $u(t) = Kx(t)$ and the matrix $A + B_1K$ is *stable* (in the Hurwitz sense), then one operates a transfer function relating v to z,

$$M(s) = (C_1 + D_1K)(sI - (A + B_1K))^{-1}B_2,$$

letting $\|H_{zv}\|_\infty = \sup_{\omega\in R} \sigma(M(j\omega)I)$, where $\sigma(M(s))$ is the largest singular number of the matrix $M(s)$. The number $\|H_{zv}\|_\infty$ defined in this way is also called the Hardy norm of the transfer function. The bound $\|H_{zv}\|_\infty < \gamma$ reflects the peculiarity of the conditional (suboptimal) H^2/H_∞ control theory in comparison with the theory of unconditional *LQR* optimization by the energy, in this case, optimality criterion.

In control theory, the concept of H_∞ control was pioneered by G. Zames; see his paper [1] published in 1981. The main results of the H^2/H_∞ control theory, first established for linear lime-invariant systems, e.g., [2, 3], were further generalized to time-varying ones and formulated in state-space terms [4, 5]. The most fruitful generalization of the H^2/H_∞ control theory was obtained within *differential game theory* [6]; subsequently, it became possible to investigate, from a uniform standpoint, time-varying deterministic and stochastic [7] systems defined on a finite horizon and systems with non-zero initial conditions for the state vector. Moreover, it became possible to study infinite-dimensional dynamic control systems and even some types of nonlinear systems. The detailed bibliography on these generalizations of the H^2/H_∞ control theory was provided in the monograph [8]; see the introduction to Section 3.2.

The transition from the theory of time-invariant systems to that of time-varying ones naturally required generalizing many concepts from the frequency-domain language to state-space terms. For example, as it turned out in several cases, the usual definition of a stable time-invariant system is convenient to be replaced by the concept of an exponentially stable system, an input-output stable system, or a system stable in some other suitable sense. Using any of these definitions, one can generalize the concepts of stabilizable, observable, etc., time-invariant system to the nonstationary case. For example, in the nonstationary case, a system $\dot{x} = A(t)x + B(t)u$ is said to be stabilizable if there exists a bounded matrix function $K(t)$ such that the system $\dot{x} = (A(t) + B(t)K(t))x$ is exponentially stable. Here is another example: given $\dot{x} = A(t)x + B(t)u$ and $y = C(t)x + D(t)v$, the pair $(A(t), C(t))$ is said to be detectable if there exists a bounded matrix function $L(t)$ such that the system $\dot{x} = (A(t) - L(t)C(t))x$ is exponentially stable. Note that in H_∞ control theory, the need for generalization also dictated the appearance of the concept of the induced H_∞ norm of an operator instead of the norm of the matrix transfer function $H_{zv}(s), Re\, s > 0$.

For a class of nondynamic causal controllers based on the system state vector $x(\cdot)$, the H^2/H_∞ control problem in the multiplicative case was solved in [9]. In other words, the controller was sought in the form $u(t) = K(t, x(\cdot)|_0^t)$, $t \geqslant 0$ and, more narrowly, in the form $u(t) = K(t)x(t)$. The controlled plant in [9] was supposed to be stochastic with the diffusion coefficient ΦdW, multiplicative by the state, control, and exogenous disturbance vectors; the case of a purely deterministic plant (with zero diffusion) was not excluded from the analysis. However, the case of simple linear diffusion of the form $(\Phi\, dW)(t) = B(t)dW(t)$ was not addressed therein. In this paper, the plant remains generally multiplicative, but the linear diffusion case is also considered. The assumptions $D_{12} = 0$, $D_{11}'C_1 = 0$, and $D_{11}'D_{11} = I$ were accepted regarding the matrix coefficients of the formula $z = C_1x + D_{11}u + D_{12}v$; in this paper, we relax these restrictions and designate D_{11} by D_1 and D_{22} by D_2 in (1.2).

There is an interesting connection between the stochastic theory of finite-dimensional multiplicative systems and the deterministic theory of matrix Lie algebras. The applications of Lie groups to ordinary differential equations are well known; for example, see [10]. But the mathematical apparatus of Lie group theory can also be used to find the fundamental matrices of stochastic multiplicative equations and integral representations for the solutions of such equations [11]. In this case, the fundamental matrix $\Phi(t, \tau)$ is a random function, and the general solution of the equation perturbed by the disturbance $h(t)$ is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau) \circ dh(\tau).$$

In this formula, $\circ dh$ denotes the Fisk–Stratonovich differential; for example, see [12].

Now we provide a small list of publications thematically close to the problem of analyzing the systems of multiplicative type under consideration. The numerical approximation of the solution of the stochastic equation

$$dx_t = (Ax_t + f(x_t))dt + \sum_{i=1}^{n}(B_ix_t + g_i(x_t))dw_i, \quad x(0) = x_0 \in R^dm$$

with nonlinear functions $f, g_i : R^d \to R^d$ was considered in [13]; by assumption, the matrices $A, B_i \in R^{d\times d}$ take values in a matrix Lie algebra $\mathfrak{g}$ with commutator relations $[A, B] = 0$, $[B_i, B_j] = 0$ for all i, j. The analysis of numerical algorithms for finding the so-called exponential integrators [14] is an active area of research in both multiplicative equations and equations with additive noise [15, 16]. The mean-square stability of numerical methods for calculating exponential integrators was investigated in [17]. Group-theoretic methods are effective for numerical integration of stochastic partial differential equations as well [18]. Among the works by Russian researchers, we note the study of an infinite-dimensional stochastic multiplicative equation with operators A and B acting in a separable Hilbert space [19]. By assumption, the operator A induces here a semigroup of operators $S(t)$, $t > 0$, of the class C_0, which guarantees the well-posedness of the Cauchy problem for the unperturbed equation $\dot{X}_t = AX(t)$.

The remainder of this paper is organized as follows. Sections 2 and 3 summarize some known results of the H^2/H_∞ control theory with state-feedback and output-feedback controllers, respectively. Section 4 is devoted to the theory of an output-feedback controller for linear time-varying stochastic systems with linear Gaussian diffusion. This is the $LEQG$ problem of risk-sensitive control, which generalizes the usual LQG problem. Section 5 presents results concerning the control of stochastic time-varying systems with a state-feedback controller; Section 6, information on the theory of multiplicative systems with a dynamic output-feedback controller. In Section 7, we describe some elements of the theory of robust stochastic systems. Concluding remarks are given in Section 8.

2. ELEMENTS OF THE H^2/H_∞ CONTROL THEORY FOR PLANTS WITH STATE-FEEDBACK CONTROLLERS

The standard problem of the deterministic H^2/H_∞ control theory of both time-invariant and time-varying control systems is formulated as follows: for a given $\gamma > 0$, find a conditionally optimal controller in the class of all admissible controllers. A controller is said to be admissible if the closed-loop system with this controller is stable and satisfies the condition $\|H_{zv}\|_\infty < \gamma$. In the simple case of a time-invariant system without control,

$$\dot{x} = Ax + Bv, \quad z = Cx, \quad x(0) = 0,$$

the admissibility requirement takes the following form [20]: the matrix A is stable and the norm of the transfer function $M(s) = C(sI - A)^{-1}B$ is bounded: $\|M(s)\|_\infty < \gamma$. According to the Kalman–Yakubovich–Popov (KYP) lemma, also known as the Bounded Real (BR) lemma in the literature [21], the admissibility condition is equivalent to each of the following two statements: (i) There exists a matrix $\tilde{P} \succ 0$ such that $A'\tilde{P} + \tilde{P}A + \tilde{P}BB'\tilde{P} + C'C \prec 0$. (ii) The Riccati equation $A'P + PA + PBB'P + C'C = 0$ has a stabilizing solution $P \succeq 0$ (i.e., the matrix $A + BB'P$ is stable).

If conditions (i) and (ii) are equivalent, then $P \prec \tilde{P}$ and, therefor, $0 \preceq P \prec \tilde{P}$. Thus, checking the controller's admissibility is reduced to checking the existence (and some properties) of the solutions of Riccati inequalities and/or equations. In the case of linear time-varying systems, the algebraic Riccati equation makes room for a differential equation, i.e., the time derivative $\dot{P}$ is added to the left-hand side of the equation for P in (ii). Consider the *game-theoretic* formulation of the conditionally optimal H^2/H_∞ control problem in the class of admissible controllers. It is based on the following observation: for the closed-loop system with an initial condition $x(0) = 0$, the constraint $\|H_{zv}\|_\infty < \gamma$ is valid iff there exists a number $\varepsilon > 0$ such that, for all $v(\cdot) \in L_2[0, \infty)$,

$$J_\gamma(u, v) := \int_0^\infty (\gamma^2\|v(t)\|^2 - \|z(t)\|^2)dt \geqslant \gamma^2\varepsilon \int_0^\infty \|v(t)\|^2 dt. \tag{2.1}$$

In the dynamic game, the first player (the controller designer) seeks to minimize the loss $J_\gamma(u, v)$ by choosing the best control $u^*(\cdot)$, whereas the second player seeks to maximize it by choosing the least favorable (worst-case) exogenous disturbance $v^*(\cdot)$. In the case $u(t) = K(t, x(\cdot)|_0^t)$, the optimal value of the payoff functional $J_\gamma(u, v)$ at the saddle point is written as $\inf_u \sup_{v \in L_2[0,\infty)} J_\gamma(u, v)$. These are the basic concepts and notation of H_∞ control theory with state-feedback controllers for deterministic (time-invariant and time-varying) systems.

The algebraic Riccati equation for P and the differential equation with the derivative $\dot{P}$ on its left-hand side will be called associated with each other. The differential equation is interesting in the sense of solutions $P(\cdot)$ for which every matrix $P(t)$ is nonnegative definite, $P(t) \succeq 0$, $t \geqslant 0$, and those for which the matrix function $t \mapsto A(t) - BB'P(t)$ is exponentially stable. It is often useful to pass to new variables in the plant's equation and assign both associated Riccati equations, algebraic and differential, to this particular state equation in the new variables. As an illustration, we give a simple example of a deterministic control system of the form

$$\dot{x} = Ax + B_1 u + B_2 \upsilon, \quad z = C_1 x + D_1 u, \quad x(0) = 0. \tag{2.2}$$

Let $G := D_1' D_1 \succ 0$; then $D_1' z = D_1' C_1 x + D_1' D_1 u$, which implies $u = \bar{u} - G^{-1} D_1' C_1 x$, where $\bar{u} := G^{-1} D_1' z$ is the new control vector. Replacing $u(\cdot)$ with $\bar{u}(\cdot)$, we write system (2.2) as

$$\dot{x} = \tilde{A} x + B_1 \bar{u} + B_2 \upsilon, \quad z = \tilde{C}_1 x + D_1 \bar{u}, \tag{2.3}$$

where $\tilde{A} = A - B_1 G^{-1} D_1' C_1$ and $\tilde{C}_1 = (I - D_1 G^{-1} D_1') C_1$. The matrix K solves the original *minimax* controller problem iff $\tilde{K} := K + G^{-1} D_1' C_1$ solves the game-theoretic problem for (2.3) with the vector $\bar{u}(\cdot)$. The pair (A, B_1) is stabilizable iff the pair $(\tilde{A}, B_1)$ is such. The main result of the application of differential game theory to the design of an H_∞ state-feedback controller for a linear time-invariant plant is formulated below (Lemma 1). Consider the Riccati differential equation on an infinite horizon ($t \in [0, \infty)$) with the condition $X(T) = M$ for some matrix $M \succeq 0$ and some $T < \infty$:

$$\begin{gathered} \dot{X} + (A - B_1 G^{-1} D_1' C_1)' X + X(A - B_1 G^{-1} D_1' C_1) \\ + C_1' (I - D_1 G^{-1} D_1') C_1 + X(B_1 G^{-1} B_1' - \gamma^{-2} B_2 B_2') X = 0. \end{gathered} \tag{2.4}$$

We denote by $X_T(t)$ the solution of equation (2.4) corresponding to $M = 0$.

Lemma 1. *Let the pair $(\tilde{A}, \tilde{C}_1)$ be detectable. Then:*

(i) For each fixed t, $X_T(t)$ is nondecreasing in T.

(ii) If there exists a solution $X \succeq 0$ of the algebraic equation with which equation (2.4) is associated, then there exists also a minimal such solution, denoted below by X^+. In addition, $X^+ \succeq X_T(t)$ for all $T \geqslant 0$. Moreover, if the pair $(\tilde{A}, \tilde{C}_1)$ is observable, then each solution $X \succeq 0$ of the algebraic equation is positive definite, $X \succ 0$.

(iii) If the solution $X^+ \succeq 0$ exists, then the controller

$$u^*(t) = K^* x(t), \quad K^* = -G^{-1}(B_1' X + D_1' C_1) \tag{2.5}$$

ensures the equality

$$\sup_{\upsilon \in L_2[0,\infty)} J_\gamma(K^* x, \upsilon) = x_0' X^+ x_0. \tag{2.6}$$

For details, we refer to [22].

3. OUTPUT-FEEDBACK CONTROLLERS WITH THE DEPENDENCE ON EXOGENOUS DISTURBANCE

Now we present the results of [21, 23] concerning the design of an H^2/H_∞-controller based on the output $y(\cdot)$ for a plant described by

$$\dot{x} = Ax + B_1 u + B_2 \upsilon, \quad z = C_1 x + D_1 u, \quad y = C_2 x + D_2 \upsilon. \tag{3.1}$$

Here, υ is an exogenous disturbance and $D_2 \neq 0$. Consider a dynamic controller with the state vector x_c of the form

$$\dot{x}_c = A_c x_c + B_c y, \quad u = C_c x_c + D_c y. \tag{3.2}$$

This controller generates an augmented system with the state vector $\bar{x} := (x',\, x_c')'$. If the plant and the dynamic controller are time-invariant systems, then the augmented system is such as well, and its transfer function $\tilde{H}_{zv}(s)$ relating v to z has the form

$$\begin{gathered}\tilde{H}_{zv}(s) = (C_1 + D_1D_cC_2 \quad D_1C_c) \\ \times \left(sI - \begin{pmatrix} A + B_1D_cC_2 & B_1C_c \\ B_cC_2 & A_c \end{pmatrix}\right)^{-1} \begin{pmatrix} B_2 + B_1D_cD_2 \\ B_cD_2 \end{pmatrix} + D_1D_cD_2.\end{gathered} \tag{3.3}$$

It generalizes the formula for the transfer function

$$H_{zv}^K(s) = (C_1 + D_1K)(sI - A - B_1K)^{-1}B_2 \tag{3.4}$$

of the closed-loop system, i.e., the plant (3.1) with the nondynamic controller $u(t) = K(t)x(t)$. The below presentation will also involve the concept of an injective mapping L of the output $y(\cdot)$, in some way dual to that of the mapping K defining the controller $u = Kx$. First, we restrict the analysis to the condition $D_c = 0$. Consider the system without control

$$\dot{x} = Ax + B_2v + Ly, \quad z = C_1x, \quad y = C_2x + D_2v. \tag{3.5}$$

The mapping L generates the transfer function

$$H_{zv}^L(s) = C_1(sI - A - LC_2)^{-1}(B_2 + LD_2)$$

of system (3.5). As mentioned in the Introduction, the mapping L is closely related to the definition of a detectable system in the nonstationary case. The introduced concepts are sufficient to formulate solvability conditions for the design problem of an output-feedback H_∞ controller.

Lemma 2. *Assume that for the plant* (3.1) *with* $D_c = 0$, *there exists a controller of the form* (3.2) *such that the state matrix* $\begin{bmatrix} A & B_1C_c \\ B_cC_2 & A_c \end{bmatrix}$ *of the augmented system is stable and the Hardy norm of the transfer function* $\tilde{H}_{zv}(s)$ *in* (3.3) *is bounded:* $\|\tilde{H}_{zv}(s)\|_\infty < 1$. *Then:*

1) There exist a matrix K of the state-feedback controller $u = Kx$ and a matrix $X \succ 0$ satisfying the Riccati equation (2.4); *in addition, the matrix $A + B_1K$ is stable and the Hardy norm of the transfer function $H_{zv}^K(s)$ in* (3.4) *is bounded, i.e.,* $\|H_{zv}^K(s)\|_\infty < 1$.

2) There exist a matrix L of the injective mapping of the output and a matrix $Y \succ 0$ such that

$$(A + LC_2)'Y + Y(A + LC_2) + YC_1'C_1Y + (B_2 + LD_2)(B_2 + LD_2)' \prec 0;$$

in addition, the matrix $A + LC_2$ is stable and the Hardy norm of the transfer function $H_{zv}^L(s)$ is bounded, i.e., $\|H_{zv}^L(s)\|_\infty < 1$. *The matrix Y satisfies the Riccati equation*

$$\begin{gathered}(A - B_2D_2'\Gamma^{-1}C_2)Y + Y(A - B_2D_2'\Gamma^{-1}C_2)' + B_2(I - D_2'\Gamma^{-1}D_2)B_2' \\ -Y(C_2'\Gamma^{-1}C_2 - \gamma^{-2}C_1'C_1)Y = 0, \quad \Gamma := D_2D_2' \succ 0.\end{gathered} \tag{3.6}$$

Note that this equation is obtained, based on the duality principle, from (2.4) by replacing the coefficients A', B_1', C_1', and D_1 with A, C_2, B_2, and D_2', respectively and, hence, replacing $G = D_1'D_1$ with $\Gamma = D_2D_2'$. Under statements 1) and 2) of Lemma 2 and additional conditions (the detectability of the pairs $(A - B_1G^{-1}D_1'C_1, (I - D_1G^{-1}D_1')C_1)$, and (A, C_2) and the stabilizability of the pairs (A, B_1) and $(A - B_2D_2'\Gamma^{-1}C_2, B_2(I - D_2'\Gamma^{-1}D_2)))$, it can be asserted [23, 24] that if the Riccati equations for X (2.4) and for Y (3.6) admit minimal solutions $X^+ \succeq 0$ and $Y^+ \succeq 0$, respectively, and $\rho(Y^+X^+) < \gamma^2$, then the dynamic game with the system of equations (3.1) and the payoff functional $J_\gamma(\cdot,\cdot)$ (2.1) has the finite value $\inf_u \sup_{\upsilon\in L_2} J_\gamma(Kx,\upsilon)$. And the optimal minimizing controller is given by the equations [22]

$$\dot{\hat{x}} = A\hat{x} + B_1u + B_2\hat{\upsilon} + (I - \gamma^{-2}Y^+X^+)^{-1}(Y^+C_2' + B_2D_2')\Gamma^{-1}(y - \hat{y}),$$
$$u^* = -G^{-1}(B_1'X^+ + D_1'C_1)\hat{x},$$

where $\hat{\upsilon} = \gamma^{-2}B_2'X^+\hat{x}$ and $\hat{y} = C_2\hat{x} + D_2\hat{\upsilon}$.

Remark. The results provided by Lemma 2 are valid for $D_c = 0$. If $D_c \neq 0$, then block (11) in the state matrix A_{cl} is replaced by $A + B_1D_cC_2$, see (3.3); for $H_{z\upsilon}$ we have $H_{z\upsilon} = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}$, where $B_{cl} = \begin{bmatrix} B_2 + B_1D_cD_2 \\ B_cD_2 \end{bmatrix}$, $C_{cl} = [C_1 + D_1D_cC_2 \;\; D_1C_c]$, and $D_{cl} = D_1D_cD_2$. Interestingly, for $D_c \neq 0$, the optimal minimizing controller coincides with the above one in the case $D_c = 0$, see [21, 23].

4. OUTPUT-FEEDBACK H_∞ CONTROLLERS FOR TIME-VARYING PLANTS WITH GAUSSIAN DIFFUSION

Systems with Gaussian (linear) diffusion are the simplest in the class of linear stochastic systems. Let

$$dx(t) = (A(t)x(t) + B_1(t)u(t))dt + B_2(t)dW(t), \quad x(0) = x_0,$$
$$dy(t) = C_2(t)x(t)dt + D_2(t)dW(t), \quad y(0) = y_0, \quad 0 \leqslant t \leqslant T$$

($(x(0), y(0))$ and $W(t)$ are often supposed to be independent). If $(x(0), y(0))$ is a Gaussian vector, then $(x(t), y(t))$ is a Gaussian process. The theory of the controller based on the output $y(\cdot)$ will be constructed starting from its interpretation as the minimax theory of output-feedback *LQG* control [25]. In this theory [26, 27], the exponent of a quadratic functional is taken as an optimality criterion:

$$J_T(u) = 2\tau \log \mathsf{E} \exp \frac{1}{2\tau}\left[x'(T)Mx(T) + \int_0^T F(x(t), u(t))dt\right],$$

where $F(x, u)$ is a quadratic form on the space of vector pairs (x, u). As usual, the problem is to find $\inf_{u(\cdot)} J_T(u(\cdot))$. This is a risk-sensitive control problem, designated as the *LEQG* problem; in the limit as $\tau \to \infty$, one obtains the standard *LQG* problem of risk-neutral control. For arbitrary τ, the controller has the form $u(t) = K\hat{x}(t)$, where $\hat{x}$ is the state vector of the filter estimating $x(t)$ from the measured output $y(\cdot)$.

We draw the reader's attention to the analogy between the results of the previous and current sections. However, note that Section 3 has presented mainly the results concerning the theory of time-invariant systems defined on an infinite horizon $0 \leqslant t < \infty$, whereas those below are related to time-invariant systems defined on $0 \leqslant t < T$, $T < \infty$. For this reason, the algebraic Riccati equations of Section 3 should be replaced by the Riccati differential equations in this section when comparing the results. Of course, a complete theory of systems of both types, deterministic and stochastic Gaussian, covers stationary and nonstationary plants.

The control problem in this section is solved under the following assumptions. First, the matrix of the quadratic form $F(x, u)$, written as $\begin{pmatrix} R(t) & \Upsilon(t) \\ \Upsilon(t)' & G(t) \end{pmatrix}$, satisfies the condition $R - \Upsilon G^{-1}\Upsilon' \succ 0$. If we define the regulated output $z = C_1x + D_1u$ and let $F(x, u) = z'z$ to clarify the analogy with the previous section results, then $R - \Upsilon G^{-1}\Upsilon' = C_1'C_1 - C_1'D_1G^{-1}D_1'C_1 \succ 0$. Suppose also that $M \succeq 0$ and $\tau = \gamma^2$. Second, we accept conditions (i)–(iii) below (see Lemma 3).

Lemma 3. *Assume that:*

(i) *The Riccati differential equation associated with the algebraic equation* (3.6), *with the initial condition* $Y(0) = Y_0$, *has a solution* $Y = Y'$ *such that* $Y(t) \succeq c_0 I$ *for some* $c_0 > 0$ *and all* $t \in [0, T]$.

(ii) *The Riccati differential equation associated with the algebraic equation* (2.4), *with the initial condition* $X(T) = M$, *has a solution* X *such that* $X(t) = X'(t) \succeq 0$ *for all* $t \in [0, T]$.

(iii) *For each* $t \in [0, T]$, $\rho(Y(t)X(t)) < \tau$, *meaning that the matrix* $I - \frac{1}{\tau}Y(t)X(t)$ *has only positive eigenvalues.*

Then the optimal control is provided by the state-feedback controller

$$u^*(t) = -G^{-1}(B_1'(t)X(t) + \Upsilon'(t))\hat{x}(t),$$

where the estimate $\hat{x}$ of the state vector x from the observed output $y(\cdot)$ is given by the *filter*

$$d\hat{x}(t) = \left(A - B_1 G^{-1}\Upsilon' - \left(B_1 G^{-1} B_1' - \frac{1}{\tau} B_2 B_2'\right) X\right) \hat{x}(t)dt$$
$$+ \left(I - \frac{1}{\tau} Y X\right)^{-1} (Y C_2' + B_2 D_2')\Gamma^{-1} \left(dy(t) - \left(C_2 + \frac{1}{\tau} D_2 B_2' X\right) \hat{x}(t)\, dt\right).$$

Lemma 3 was established in [22, 28].

A new aspect of the theory is the need to answer the following question: What is the class of admissible controllers in this theory? It is required that in the notation $e(t) := x(t) - \hat{x}(t)$, $\epsilon(t) := e(t) + \frac{1}{\tau} Y X \hat{x}(t)$, the process

$$\alpha(t) := B_2' Y \epsilon(t) - D_2' \Gamma^{-1}(C_2 Y + D_2 B_2') Y^{-1} e(t)$$

would generate the exponent

$$\zeta(t) = \exp\left\{ \int_0^t \alpha'(s) dW(s) - \frac{1}{2} \int_0^t \|\alpha(s)\|^2 ds \right\},$$

which is known to be a martingale on $[0, T]$; see [29]. This condition holds at least for the linear controller [28]. Then it is easy to check the following fact: in the class of admissible controllers, the infimum of the optimality criterion is ensured by the linear controller.

Also, some comments are needed to generalize the results of this section to the case of *infinite-horizon* systems ($T = \infty$). The coefficients in the plant's equations at $T = \infty$ should be supposed to be independent of t, the condition $y(0) = 0$ should be imposed, and the cost functional should be defined as $J(u) := \lim_{T\to\infty} \frac{1}{T} J_T(u)$. Next, the Riccati differential equations should be replaced by the algebraic ones, and assumptions (i)–(iii) of Lemma 3 should be further specified by replacing the existence condition for a symmetric solution Y with two conditions:

(a) $R - \Upsilon G^{-1} \Upsilon' \succeq 0$.

(b) The pair of matrices $(A - B_1 G^{-1} \Upsilon', R - \Upsilon G^{-1} \Upsilon')$ is detectable, and the pair $(A - B_2 D_2' \Gamma^{-1} C_2, B_2(I - D_2' \Gamma^{-1} D_2))$ is stabilizable.

In addition, the designations X and Y in Lemma 3 should be replaced by the commonly used X_∞ and Y_∞, and the following assumptions should be made:

(i)$'$ The equation for Y admits a minimal solution $Y_\infty \succ 0$.

(ii)$'$ The equation for X admits a minimal solution $X_\infty \succeq 0$.

Then it is proved that X_∞ and Y_∞ can be obtained from $X = X_T$ and $Y = Y_T$, respectively, by passing to the limit as $T \to \infty$. This result constitutes the next lemma.

Lemma 4. *Under the above conditions, the following statements are true:*

(i) For $M = 0$, the differential equation associated with (2.4) has a solution $X_T(t) \succeq 0$ such that $X_T(t) \to X_\infty$ as $T \to \infty$.

(ii) If $Y_\infty \succeq Y_0 \succ 0$, then for $M = 0$, the differential equation associated with (3.6) has a solution $Y(t; Y_0)$ such that $Y(t; Y_0) \to Y_\infty$ as $t \to \infty$.

(iii) For each $T > 0$ and $t \in [0, T]$, the matrices $Y(t; Y_0)$ and $X_T(t; M)$ satisfy the inequality $\rho(Y(t)X(t)) < \tau$.

See [22, 28].

Lemma 4 is proved as follows. First, we check that for $M = 0$, the solution $X_T(t) \succeq 0$ of equation (2.4) converges to X_∞ as $T \to \infty$. To verify statement *(ii)* of Lemma 4, we consider the system

$$\dot{\eta} = A'\eta + C_2'\nu + R^{1/2}\omega, \quad \eta(0) = \eta_0, \quad \zeta = B_2'\eta + D_2'\nu,$$

where $\eta \in R^n$ is the state vector, $\nu \in R^l$ is the control input, $\omega \in R^q$ is a disturbance, and $\zeta \in R^p$ is the regulated output, and define the functionals

$$\begin{aligned} \bar{J}_\tau^{T,Y_0}(\nu, \omega) &:= \eta(T)'Y_0\eta(T) + \int_0^T (\|\zeta(t\|^2 - \tau\|\omega(t)\|^2)dt, \\ \bar{J}_\tau(\nu, \omega) &:= \int_0^\infty (\|\zeta(t)\|^2 - \tau\|\omega(t)\|^2)\, dt,\ \omega \in L_2[0, T]. \end{aligned} \tag{4.1}$$

Applying then differential game theory, we conclude that $\inf_\nu \sup_\omega \bar{J}_\tau(\nu, \omega) < \infty$ on an infinite horizon and there exists the limit Y_∞ for equation (3.6) with the solutions Y_T for each $T < \infty$. Thus, equation (3.6) has a minimal solution $Y_\infty \succ 0$, and by Lemma 1 of Section 2,

$$\inf_\nu \sup_\omega \bar{J}_\tau(\nu, \omega) = \eta_0' Y_\infty \eta_0. \tag{4.2}$$

Finally, since the variables X^+ and Y_∞ are dual, the formula $\nu^* = -\Gamma^{-1}(D_2B_2' + C_2Y_\infty)\eta$ is dual to formula (2.5), written as $u^* = K^*x(t)$, where $K^* = -G^{-1}(D_1'C_1 + B_1'X^+)$. Similarly, formulas (2.6) and (4.2) are dual as well. However, the inequality $Y_\infty \succeq Y_0$ has no analog in Lemma 1. But formula (4.1) and $Y_0 \preceq Y_\infty$ obviously imply $\bar{J}_\tau^{T,Y_0} \preceq \bar{J}_\tau^{T,Y_\infty}$ for $\nu = \nu^*$, and we ultimately arrive at (4.2). See [22, 28].

5. STATE-FEEDBACK H_∞ CONTROLLERS FOR UNCERTAIN STOCHASTIC SYSTEMS

This section considers a general multiplicative stochastic system with a state-feedback controller. The controlled plant is described by equations (1.1) and (1.2); the controller, by the formula $u(t) = Kx(t)$; the closed-loop system state, by the equation

$$dx = ((A + B_1K)x + B_2v)dt + A_0xdw_0 + B_{01}Kxdw_1 + B_{02}vdw_2. \tag{5.1}$$

System (5.1) is uncertain due to the dependence of its dynamics on the unknown parameter K to be found. The coefficient at x in the diffusion component of this equation, equal to $A_0dw_0 + B_{01}Kdw_1$, can be written as

$$([A_0\ 0] + [0\ B_{01}]K)dW_1, \quad dW_1 := \begin{pmatrix} dw_0 \\ dw_1 \end{pmatrix}.$$

Thereby, we replace the three Wiener processes w_0, w_1, and w_2 in (5.1) with the two processes W_1 and w_2. In other words, keeping the same notation for the matrix coefficients, it is possible to let $w_0 = w_1$ in (5.1):

$$dx = ((A + B_1K)x + B_2v)dt + (A_0 + B_{01}K)xdw_1 + B_{02}vdw_2. \tag{5.2}$$

Further, without loss of generality, we take

$$z(t) = C_1x(t) + D_1u(t), \quad y(t) = C_2x(t) + D_2v(t)$$

instead of the pair of equations (1.2). For $t \geqslant s$, where s is the initial time instant, the closed-loop system

$$dx = (A + B_1K)xdt + (A_0 + B_{01}K)xdw_1, \quad x(s) = h, \tag{5.3}$$

obtained from (5.2) with $v(t) \equiv 0$, will be called nominal. This system is time-varying even under constant matrix coefficients in (5.2), and its stability should be understood, e.g., as exponential stability in mean square if the solution $x(t)$ is desired to be an element of the space $L_2(s, \infty)$ of all functions for which $\int_s^\infty \|x(t)\|^2 dt < \infty$. In the nonstationary case, one should find a suitable analog of the boundedness $\|H_{zv}\|_\infty < \gamma$ of the induced norm of the operator H_{zv}. The following condition is a suitable stochastic analog of norm boundedness: there exists a constant $\epsilon > 0$ such that for $x(s) = 0$,

$$\mathsf{E}\int\limits_s^\infty \left(\|(C_1 + D_1K)x(t)\|^2 - \gamma^2\|v(t)\|^2\right) dt \leqslant -\epsilon\gamma^2\mathsf{E}\int\limits_s^\infty \|v(t)\|^2 dt \tag{5.4}$$

for each $v \in L_2(s, \infty)$. Note that the exponential stability of the nominal system is a *sufficient* condition for equation (5.2) to have solutions belonging to $L_2(s, \infty)$ for any $v \in L_2(s, \infty)$.

Now we formulate the stochastic H_∞ controller design problem solved in this section: *for system* (5.2)*, find a matrix K such that the nominal system* (5.3) *is exponentially stable in mean square, and the closed-loop system* (5.2) *satisfies the requirement* (5.4) *of the H_∞ boundedness of the norm of the transfer operator H_{zv}*. As in Section 2, the most fruitful approach for the stochastic case is to apply the theory of linear-quadratic differential games associated with the Itô multiplicative equation and the functional $J(u, v) = \int_s^\infty \mathsf{E}(z'(t)z(t) - \gamma^2\|v(t)\|^2)dt$. Here, the bridge between the theory of H_∞ controllers and game theory is the *stochastic BR* lemma; see the presentation below, following the monograph [8].

Suppose that the system $dx(t) = Ax(t)dt + A_0x(t)dw_1(t)$, $x(s) = h$, corresponding to the choice $u(\cdot) \equiv 0$ and $v(\cdot) \equiv 0$ in equation (5.2), is exponentially stable (and hence the matrix A is stable) and there exists a constant ϵ_2 such that $J_2(0, v) \leqslant -\epsilon_2 \int_0^\infty \|v(t)\|^2 dt$ for all $v \in L_2(0, \infty)$. Then the following stochastic BR lemma is true.

Lemma 5. *Part 1* (*existence*):

(*a*) *For each $s \geqslant 0$ and $h \in L_2(\Omega, \mathcal{F}_s, P)$, there exists a unique disturbance $v_2^s(\cdot) \in L_2(s, \infty)$ such that $J(0, v_2^s(\cdot)) = \sup_{v \in L_2(s,\infty)} J(0, v)$.*

(*b*) *There exists a matrix $X_2 \succeq 0$ such that* $\sup_{v \in L_2(s,\infty)} J(0, v) = \mathsf{E}\langle h, X_2h\rangle$.

(*c*) *For any $T > 0$, given $X_{2T}(T) = 0$, there exists a unique solution $X_{2T}(\cdot) \succeq 0$ of the generalized Riccati equation*

$$\dot{X}_{2T} + A'X_{2T} + X_{2T}A + A_0'X_{2T} + R + X_{2T}B_2(I - B_{02}'X_{2T}B_{02})^{-1}B_2'X_{2T} = 0.$$

Part 2 (minimax solution):

(*d*) *The worst-case disturbance* $\upsilon^s_{2T}(\cdot)$ *maximizing the functional*

$$J_T(u,\upsilon) = \int\limits_s^T \mathsf{E}(z'(t)z(t) - \gamma^2\|\upsilon(t)\|^2)dt, \quad s \leqslant T < \infty,$$

has the form

$$\upsilon^s_{2T} = (I - B'_{02}X_{2T}(t)B_{02})^{-1}B'_2X_{2T}(t)x^s_{2T}(t),$$

where $x^s_{2T}(\cdot)$ *is the optimal trajectory representing the solution of the closed-loop system*

$$\begin{aligned} dx(t) = {}& (A + B_2(I - B'_{02}X_{2T}(t)B_{02})^{-1}B'_2X_{2T}(t))x(t)dt + A_0x(t)dw_1(t) \\ & + B_{02}(I - B'_{02}X_{2T}(t)B_{02})^{-1}B'_2X_{2T}(t)x(t)dw_2(t), \quad x(s) = h. \end{aligned}$$

(*e*) *The matrix* X_2 *in item* (*b*) *is also the minimal solution of the* generalized *Riccati equation*

$$A'X_2 + X_2A + A'_0X_2A_0 + R + X_2B_2(I - B'_{02}X_2B_{02})^{-1}B'_2X_2 = 0,$$

with the property $I - B'_{02}X_2(t)B_{02} \succ 0$.

We formulate the main theorem of Section 5 for the case of a multiplicative system on an infinite horizon.

Theorem 1. *Under the assumptions of the stochastic BR lemma, let* $\|C_1x + D_1u\|^2 > \epsilon_1\|u\|^2$ *for some* $\epsilon_1 > 0$ *and all* $x \in R^n$, $u \in R^m$. *Then:*

(*a*) *For* $h \in L_2(\Omega, \mathcal{F}_s, P)$ *and all* $s \geqslant 0$, *there exists a unique minimax pair for the functional* $J(u,\upsilon)$.

(*b*) *There exists a unique solution* $X \succeq 0$ *of the algebraic Riccati equation*

$$\begin{aligned} & A'X + XA + A'_0XA_0 + R + XB_2(I - B'_{02}XB_{02})^{-1}B'_2X \\ & - (XB_1 + A'_0XB_{01} + Q)(G + B'_{01}XB_{01})^{-1}(XB_1 + A'_0XB_{01} + Q)' = 0 \end{aligned} \tag{5.5}$$

such that $I - B'_{02}XB_{02} \succ 0$ *and* $V = \mathsf{E}\langle h, Xh\rangle$.

(*c*) *The minimax pair is given by* $u = F_1x$, $\upsilon = F_2x$, *where*

$$\begin{aligned} F_1 &= -(G + B'_{01}XB_{01})^{-1}(XB_1 + A'_0XB_{01} + Q)', \\ F_2 &= (I - B'_{02}XB_{02})^{-1}B'_2X). \end{aligned} \tag{5.6}$$

(*d*) *The closed-loop stochastic system*

$$dx = (A + B_1F_1 + B_2F_2)xdt + (A_0 + B_{01}F_1)xdw_1(t) + B_{02}F_2xdw_2(t) \tag{5.7}$$

with $x(s) = h$ *is exponentially stable in mean square.*

It seems interesting to compare the conclusions in Theorem 1 with the results of solving the same problem on a finite horizon [30]. Let

$$J_T(u,\upsilon) = \int\limits_s^T \mathsf{E}(z'(t)z(t) - \gamma^2\|\upsilon(t)\|^2)dt, \quad s \leqslant T < \infty.$$

Consider the standard stochastic control problem $\inf\limits_{u\in L_2[s,T]} J_T(u,0)$. The solution $X_{1T} \succeq 0$ of the generalized Riccati equation

$$\begin{aligned}&\dot X_{1T} + A'X_{1T} + X_{1T}A + A_0'X_{1T}A_0 + R - (X_{1T}B_1 + A_0'X_{1T}B_{01} + Q)\\ &\times (G + B_{01}'X_{1T}B_{01})^{-1}(X_{1T}B_1 + A_0'X_{1T}B_{01} + Q)' = 0, \quad X_{1T}(T) = 0,\end{aligned} \tag{5.8}$$

is associated with this problem. The equation has a unique solution $X_{1T} \succeq 0$ determining the optimal controller

$$u_{1T} = -(G + B_{01}'X_{1T}B_{01})^{-1}(B_1'X_{1T} + B_{01}'X_{1T}A_0 + Q')x.$$

The solutions X_{1T} and X_{2T} of both Riccati differential equations allow solving the minimax control problem with the general criterion $J_T(u,\upsilon)$ on a *finite horizon*. The following result was established in [8].

Theorem 2. *Let $F(x,u) > \epsilon_1\|u\|^2$ and $J(0,\upsilon) \leqslant -\epsilon_2\int_0^\infty \|\upsilon(t)\|^2dt$ for all $\upsilon \in L_2(0,\infty)$. Then the game-theoretic minimax problem with the criterion $J_T(u,\upsilon)$ has a saddle point $(u_T(\cdot),\upsilon_T(\cdot))$. The Riccati differential equation*

$$\begin{aligned}&\dot X_T + A'X_T + X_TA + A_0'X_TA_0 + R - (X_TB_1 + A_0'X_TB_{01} + Q)(G + B_{01}'X_TB_{01})^{-1}\\ &\times(X_TB_1 + A_0'X_TB_{01} + Q)' + X_TB_2(I - B_{02}'X_TB_{02})^{-1}B_2'X_T = 0, \quad X_T(T) = 0\end{aligned} \tag{5.9}$$

has a unique solution $X_T \succeq 0$. The saddle point of the game is given by $u_T = F_{1T}x$, $\upsilon_T = F_{2T}x$, where

$$\begin{aligned}F_{1T} &= -(G + B_{01}'X_TB_{01})^{-1}(B_1'X_T + B_{01}'X_TA_0 + Q'),\\ F_{2T} &= (I - B_{02}'X_TB_{02})^{-1}B_2'X_T.\end{aligned}$$

The proof of this theorem is quite complicated, it uses results from several works [31–33].

It is interesting to compare the Riccati equations (5.6) and (5.5): the last term on the right-hand side of (5.6) vanishes in (5.5). Of course, this is because equation (5.5) is associated with the criterion $J_T(u,0)$ whereas equation (5.6) with the criterion $J_T(u,\upsilon)$. The appearance of the term $X_TB_2(I - B_{02}'X_TB_{02})^{-1}B_2'X_T)^{-1}$ seems quite natural in the problem of maximizing the functional $J_T(u,\upsilon)$ with respect to the second argument.

Also, it is interesting to represent the criterion $J_T(u,\upsilon)$ through the functions u_T and υ_T determining the saddle point $(u_T(\cdot),\upsilon_T(\cdot))$. Suppose that a solution X_T of equation (5.6) exists on the closed interval $[T-\alpha,T]$, and let s be a point of this interval. Then, applying Itô's formula to the quadratic form $x(t)'X_Tx(t)$, where $x(t)$ satisfies equation (1.1) under the constraint $w_0 = w_1$, we obtain

$$\begin{aligned}J_T(u,\upsilon) = \mathsf{E}\int_s^T &\|(I - B_{02}'X_TB_{02})^{\frac12}(\upsilon(t) - F_{2T}x(t))\|^2dt\\ &+ \mathsf{E}\int_s^T \|(G + B_{01}'X_TB_{01})^{\frac12}(u(t) - F_{1T}x(t))\|^2dt.\end{aligned}$$

Therefore, the saddle point satisfies the condition

$$J_T(u_T,\upsilon) \leqslant J_T(u_T,\upsilon_T) = \mathsf{E}h'X_T(s)h \leqslant J_T(u,\upsilon_T).$$

For a sufficiently small parameter α, this condition is used to prove the existence of a solution of the Riccati differential equation (5.6) on a finite interval $t \in [s,T]$.

6. OUTPUT-FEEDBACK H_∞ CONTROLLERS FOR STOCHASTIC SYSTEMS WITH MULTIPLICATIVE DISTURBANCES

Consider an Itô stochastic differential equation of the form

$$\Sigma: \begin{cases} dx(t) = (Ax(t) + B_1 \upsilon(t) + B_2 u(t))dt + A_0 x(t) dw_0(t) \\ \qquad\qquad + B_{01}\upsilon(t)dw_1(t) + B_{02}u(t)dw_2(t),\ x_0 \in \mathrm{R}^n \\ z(t) = C_1 x(t) + D_{11}\upsilon(t) + D_{12}u(t) \\ y(t) = C_2 x(t) + D_{21}\upsilon(t), \quad t \in [0, T]. \end{cases} \tag{6.1}$$

The controller in the feedback loop is a deterministic *dynamic* system with a state vector $\hat{x}$, described by the equations

$$d\hat{x}(t) = A_k \hat{x}(t)dt + B_k\, y(t)dt, \quad u(t) = C_k \hat{x}(t) + D_k\, y(t), \tag{6.2}$$

where all matrix coefficients are to be determined.

We denote by $\bar{x}$ the state vector of the closed-loop system and let $\bar{x}' := (x', \hat{x}')$. The stochastic equation for $\bar{x}$ is obtained by some cumbersome, albeit elementary, calculations. As is easily verified, the functions $t \mapsto \bar{x}(t),\ t \mapsto z(t)$ must satisfy the equations

$$\Sigma_2: \begin{cases} d\bar{x}(t) = A_{cl}\bar{x}(t)\, dt + B_{cl}\upsilon(t)\, dt + A^0_{cl}\bar{x}(t)dw_0(t) \\ \qquad\qquad + B^0_{cl}\upsilon(t)dw_1(t) + A^1_{cl}\bar{x}(t)\, dw_2(t) + B^1_{cl}\upsilon(t)\, dw_2(t) \\ z(t) = C_{cl}\bar{x}(t) + D_{cl}\upsilon(t), \quad t \in [0, T]. \end{cases} \tag{6.3}$$

The formulas expressing the coefficients of the equations of the closed-loop system Σ_2 are directly derived; for example, see [34].

The main result of the theory of the dynamic output-feedback controller is provided immediately below.

Theorem 3. *For system* (6.1) *and* $\gamma > 0$, *the following statements are equivalent:*

(*i*) *There exists a controller* (6.2) *such that the corresponding closed-loop system* (6.3) *is internally stable and* $\|H_{z\upsilon}\|_\infty < \gamma$.

(*ii*) *There exists a matrix function* $P: t \mapsto P(t) \prec 0$ *such that* $\mathcal{M}(\gamma, P) \succ 0$ *for all* $t \in [0, T]$.

See [7, Theorem 3.3]. Here, $\mathcal{M}(\gamma, P)$ is the block matrix of dimensions 2×2 for a quadratic form in the space of vector pairs $\binom{\bar{x}}{\upsilon}$.

For the further presentation, it is convenient to write the condition $\mathcal{M}(\gamma, P) \succ 0$ in equivalent form, replacing the block diagonal matrix $\operatorname{diag}\{\mathcal{M}(\gamma, P), I\}$ with a block matrix $T\mathcal{N}(\gamma, P)T' \succ 0$ of dimensions 3×3, where

$$\mathcal{N}(\gamma, P) = \begin{pmatrix} PA_{cl} + A'_{cl}P + S_{11} & PB_{cl} + S_{12} & C'_{cl} \\ B'_{cl}P + S_{21} & \gamma^2 I + S_{22} & D'_{cl} \\ C_{cl} & D_{cl} & I \end{pmatrix}, \quad T = \begin{pmatrix} I & O & -C'_{cl} \\ O & I & -D'_{cl} \\ O & O & I \end{pmatrix}.$$

The matrices S_{ij} were found in [34]. For the blocks (submatrices) N_{ij} of the nonnegative definite matrix $\mathcal{N}(\gamma, P)$, the formulas presented therein are

$$N_{11} = P(A^0 + B^I M_k C^I) + (A^0 + B^I M_k C^I)'P + S_{11},$$
$$N_{12} = P(B^0 + B^I M_k D^0_{21} + S_{12}), \quad N_{13} = (C^0 + D^0_{12} M_k C^I)',$$
$$N_{22} = \gamma^2 I + S_{22}, \quad N_{23} = (D_{11} + D^0_{12} M_k D^0_{21})', \quad N_{33} = I,$$

where $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ is the parameter matrix of the dynamic controller. The matrices A_{cl}, B_{cl}, C_{cl}, and D_{cl} are expressed affinely through the matrix M_k :

$$A_{cl} = A^0 + B^I M_k C^I, \quad B_{cl} = B^0 + B^I M_k D_{21}^0,$$
$$C_{cl} = C^0 + D_{12}^0 M_k C^I, \quad D_{cl} = D_{11} + D_{12}^0 M_k D_{21}^0,$$

with some matrix coefficients [30]. We represent the matrix $\mathcal{N}(\gamma, P)$ as the sum of a matrix $\mathcal{H}$, independent of M_k, and two matrices that linearly depend on M_k. As a result, $\mathcal{H}+\mathcal{Q}'M_k'\mathcal{R}+\mathcal{R}'M_k\mathcal{Q}$ with some matrices $\mathcal{Q}$ and $\mathcal{R}$ and a nonnegative definite matrix $\mathcal{H}$. According to the stochastic generalization [7] of the *projection lemma* from the theory of linear matrix inequalities (LMIs, see [35]), the *LMI*

$$\mathcal{H} + \mathcal{Q}'M_k'\mathcal{R} + \mathcal{R}'M_k\mathcal{Q} \succ O$$

has a solution M_k iff the matrix $\mathcal{H}$ is positive definite on the null subspaces $\ker\mathcal{Q}$ and $\ker\mathcal{R}$ of the matrices $\mathcal{Q}$ and $\mathcal{R}$, respectively.

The projection lemma gives a necessary and sufficient condition for $\|H_{zv}\|_\infty < \gamma$. The condition is formulated in terms of LMIs instead of matrix differential equations. This lemma settles the issue about the admissibility conditions of the controller M_k and, moreover, allows calculating the parameters A_k, B_k, C_k, and D_k of the controller if they are unknown [7].

7. STOCHASTIC ROBUST ANALYSIS OF THE SYSTEM WITH PARAMETRIC PERTURBATION

Let

$$dx(t) = Ax(t)dt + A_0x(t)dw_1(t), \quad 0 < t < T, \tag{7.1}$$

be a nominal stochastic system and

$$dx(t) = (A + B\Delta C)x(t)dt + A_0x(t)dw_1(t) + B_0\Delta Cx(t)dw_2(t) \tag{7.2}$$

be its *stochastic* disturbance with a simultaneous *parametric* perturbation of the matrix parameter A. In system (7.2), the perturbing parameter is an arbitrary matrix Δ from the set $\mathcal{D} = R^{l\times q}$ of all matrices of dimensions $l \times q$. Being time-varying, the nominal system (7.1) is assumed to be stable in the following sense: there exists a constant $c > 0$ such that $\mathsf{E}\int_0^\infty \|x(t)\|^2dt \leqslant c\|x^0\|^2$, where $x(\cdot) = x(\cdot, x^0)$ is the trajectory of equation (7.1) starting at $x^0 \in R^n$. The Wiener processes w_1 and w_2 are independent, the perturbing force is a state-multiplicative process, and the uncertainty of system (7.2) is measured by the norm $\|\Delta\|$ of the matrix Δ.

Assume that $\rho > 0$ is a small number. For small $\|\Delta\| < \rho$, system (7.2) is close to the unperturbed one (7.1) and is probably stable as well. What is the value $\rho_{\max}$ of the parameter ρ under which the stability of system (7.2) is ensured for each Δ from the set $\mathcal{D} = \{\Delta : \|\Delta\| < \rho_{\max}\}$? It is natural to call the number $\rho_{\max}$ the *robust* stability radius of system (7.1) with respect to uncertainties $\Delta \in \mathcal{D}$. Accordingly, we introduce the following definition: the number $r_{\mathcal{D}} = \inf\{\|\Delta\| :$ system (7.2) with the uncertainty Δ is unstable$\}$ is called the robust stability radius of system (7.1).

Below, to relate robust stability to H_∞ control analysis (based on the stochastic BR lemma), let us address the standard plant of the bilinear stochastic H_∞ control theory:

$$\begin{cases} dx(t) = Ax(t)dt + Bv(t)dt + A_0x(t)dw_1(t) + B_0v(t)dw_2(t) \\ z(t) = Cx(t). \end{cases} \tag{7.3}$$

Interpreting υ as a control action, we consider a block with the input z and output $\upsilon = \Delta z$ in the feedback loop. Then the closed-loop system becomes the perturbed system (7.2). In H_∞ control theory, the operator associated with this system is a disturbance operator $L : \upsilon \mapsto z$, specifying the effect of an exogenous disturbance (here, control) υ on the output z. The operator L acts from the space of functions $\upsilon(\cdot)$ into the space of functions $z(\cdot) = Cx(\cdot)$, where $x(\cdot) = x(\cdot, \upsilon, x^0)|_{x^0=0}$. Thus, $L : \upsilon(\cdot) \mapsto Cx(\cdot, \upsilon, 0)$.

The above definition of the robust stability radius is generalized to nonstationary and nonlinear uncertainties. For each $t \in R_+$, let $\Delta(t, \cdot)$ be a linearly bounded mapping $R^q \to R^l$, i.e., $\|\Delta(t, y)\| \leqslant K\|y\|$ for some $K > 0$ and all $t \in R_+$ and $y \in R^q$. In addition, assume that this mapping is Lipschitz bounded: for any $T > 0$ there exists a constant $L(T)$ such that $\|\Delta(t, y_1) - \Delta(t, y_2)\| \leqslant L(T)\|y_1 - y_2\|$ for all $y_1, y_2 \in R^q$ and all $t \in [0, T]$. The uncertainty Δ of this kind is nonlinear and nonstationary. Its value is found as the smallest K in the definition of linear boundedness, and the equation associated with such uncertainty has the form

$$dx(t) = (Ax(t) + B\Delta(t, Cx(t)))dt + A_0x(t)dw_1(t) + B_0\Delta(t, Cx(t))dw_2(t). \tag{7.4}$$

Let $\mathcal{D}_{tn}$ be the set of all such uncertainties. We denote by $x_\Delta(\cdot, x^0)$ the solution of equation (7.4), which is assumed to be unique in the class of random functions $L^2([0, T]; L^2(\Omega, R^n))$, and call system (7.4) stable if its solutions satisfy the condition $\int_0^\infty \mathsf{E}\|x_\Delta(t, x^0)\|^2 dt \leqslant c\|x^0\|^2$ for some constant c. In H_∞ control theory, it is required to ensure the inequality $\|L_{z\upsilon}\| < \gamma$; in robust systems theory, the inequality $\|\Delta_{\upsilon z}\| < \rho$. We choose γ as small as possible and ρ as large as possible.

Due to the linear and Lipschitz boundedness conditions of the function Δ, equation (7.4) has a unique solution $x_\Delta(\cdot, x^0)$, which is a stochastic process with bounded second moments [36]. In integral form, equation (7.4) is written as

$$x_\Delta(t) = x^0 + \int_0^t (Ax_\Delta(s) + B\upsilon_\Delta(s))ds + \int_0^t [A_0x_\Delta(s)\ B_0\upsilon_\Delta(s)]dw(s), \quad t \in [0, T],$$

for each $T > 0$, where $\upsilon_\Delta(\cdot) = \Delta(\cdot, Cx_\Delta(\cdot)), w(s) = [w_1(s),\, w_2(s)]'$. Let $\|\Delta\| < \|L_{z\upsilon}\|^{-1}$, then there exists a number $\gamma > \|L_{z\upsilon}\|$ such that $\gamma\|\Delta\| < 1$. In H_∞ control theory, the functional

$$J_T(x^0, \upsilon) = \int_0^T \mathsf{E}[\gamma^2\|\upsilon(t)\|^2 - \|z(t)\|^2]dt$$

is given by the well-known formula

$$J_T(x^0, \upsilon) = \langle x^0, P(0)x^0\rangle - \mathsf{E}\langle x(T), P(T)x(T)\rangle + \int_0^T \mathsf{E}\left\langle \begin{pmatrix} x(t) \\ \upsilon(t) \end{pmatrix}, M(P(t)) \begin{pmatrix} x(t) \\ \upsilon(t) \end{pmatrix} \right\rangle dt. \tag{7.5}$$

According to the stochastic BR lemma, for any $\gamma > 0$ the following statements are equivalent:

(i) There exists a matrix $P \prec 0$ such that $M(P) \succ 0$.

(ii) The equation for $x(\cdot)$ is internally stable, with $\|L_{z\upsilon}\| < \gamma$.

The relation $M(P) \succ 0$ implies $M(P) \succeq \delta^2 I$ for some number $\delta > 0$. Calculating $J_T(x^0, \upsilon)$ (7.5) for $x(\cdot) = x_\Delta(\cdot, x^0)$ and $\upsilon(\cdot) = \upsilon_\Delta(\cdot)$ yields

$$J_T(x^0, \upsilon_\Delta) = \int_0^T \mathsf{E}[\gamma^2\|\Delta(t, Cx_\Delta(t))\|^2 - \mathsf{E}\|Cx_\Delta(t)\|^2]dt. \tag{7.6}$$

To proceed, we formulate an important result on the stability of the uncertain system (7.4).

Theorem 4. *Assume that system* (7.1) *with* $\Delta = 0$ *is stable and the uncertainty* $\Delta, \Delta \neq 0$, *satisfies the condition* $\|\Delta\| = \sup\{\|\Delta(t, y)\| \diagup \|y\| : t > 0, y \in R^q, y \neq 0\} < \|L\|^{-1}$, *where* $L = L_{zv}$. *Then the perturbed* (*uncertain*) *system* (7.4) *is stable. In particular,* $r_{\mathcal{D}_{tn}} \geqslant \|L\|^{-1}$.

Indeed, since

$$\gamma^2\|\Delta(t, Cx_\Delta(t)\|^2 \leqslant \gamma^2\|\Delta\|^2\|Cx_\Delta(t)\|^2 \leqslant \gamma^2\|Cx_\Delta(t)\|^2$$

by the condition $\|L\|\|\Delta\| < \gamma\|\Delta\| < 1$, from (7.6) it follows that $J_T(x^0, \upsilon_\Delta) \leqslant 0$. By analogy, letting $x(\cdot) = x_\Delta(\cdot, x^0)$, $\upsilon(\cdot) = \upsilon_\Delta(\cdot)$, and $M(P) \succeq \delta^2 I$, we estimate the integral in (7.5) as

$$\int_0^T \delta^2 \mathsf{E}\{\|x_\Delta(t)\|^2 + \|\upsilon_\Delta(t)\|^2\}dt \geqslant \int_0^T \delta^2 \mathsf{E}\|x_\Delta(t)\|^2 dt.$$

Thus, for $x(\cdot) = x_\Delta(\cdot, x^0)$ and $\upsilon(\cdot) = \upsilon_\Delta(\cdot)$, formula (7.5) leads to the inequality

$$\mathsf{E}\langle x_\Delta(T), (-P)x_\Delta(T)\rangle \leqslant \langle x_0, (-P)x_0\rangle - \int_0^T \delta^2 \mathsf{E}\|x_\Delta(t)\|^2 dt$$

for any $T > 0$. The left-hand side of this inequality is positive due to $-P \succ 0$; therefore, $\int_0^T \delta^2 \mathsf{E}\|x_\Delta(t)\|^2 dt \leqslant \|P\|\|x^0\|^2/\delta^2$. This proves the stability of system (7.4).

As is easily verified, the perturbed closed-loop system obtained by substituting $\upsilon = \Delta z$ into equation (7.3) can be reduced to

$$d\bar{x}(t) = (A_{cl} + B_{cl}\Delta C_{cl})\bar{x}(t)dt + \left(A_{cl}^0 dw_1(t) + B_{cl}^0 \Delta C_{cl} dw_2(t)\right)\bar{x}(t),$$

where $\bar{x} := (x, \hat{x})$. Its coefficients with the *cl* subscript are calculated straightforwardly. In combination with Theorems 3 and 4, this result gives the following fact.

Lemma 6. *Let* γ_{opt} *be the infimum of those* $\gamma \geqslant 0$ *for which there exists a dynamic controller ensuring the internal stability of the closed-loop system* (7.4) *and the condition* $\|L_{cl}\| < \gamma$. *Then for any* $\gamma > \gamma_{opt}$, *there exists a controller* Δ *such that the corresponding closed-loop system* (7.4) *has a robust stability radius* $r_{\mathcal{D}}(A_{cl}, B_{cl}, C_{cl}, A_{cl}^0, B_{cl}^0) > \gamma^{-1}$.

For details, we refer to [7].

8. CONCLUSIONS

The topic of this paper has been the H^2/H_∞ branch of general control theory for technical systems (plants). This branch deals with, first, the problems of controller *analysis* and, second, the problems of controller *design*, i.e., optimization in the class of *admissible* controllers identified at the analysis stage. Note that H_∞ control theory solves the problem of rejecting an exogenous disturbance acting on a plant in the closed loop with an admissible controller. An admissible controller ensures that, first, the closed-loop system (plant + controller) is stable and, second, the H_∞ norm of the transfer operator relating an exogenous disturbance υ to a regulated output z is below a threshold $\gamma > 0$, set a priori by the designer: $\|H_{z\upsilon}\|_\infty < \gamma$. The controller is defined as a functional mapping $K : y \mapsto u$ of an observed output y (measured by a noisy sensor) into a control action u. In this problem statement, the system must have two inputs υ and u as well as two outputs z and y. Further, the system must be stochastic, given by an Itô stochastic equation whose diffusion term is not arbitrary but of a partial (multiplicative) structure; however, it is not the same as the structure in the linear Itô equation. The stochastic nature of the system is due to that of, first, the exogenous disturbance and /or observed output and, second, the nominal (unperturbed) system with zero exogenous disturbance. The standard model of a stochastic system in H_∞ control theory is a perturbed system. And the nominal system is perturbed by two stochastic forces: the force $B_1 v dt$ (which may be deterministic) and the stochastic force $B_0 v dw_2$. The presence of both types of disturbances is an essential point of the most complete generalization of H_∞ control theory. This is the case for the H_∞ controller design problem in the statistical H^2/H_∞ control theory.

On the other hand, robust control theory deals with uncertain systems, their robust stability, and calculation of the stability radius of closed-loop systems. The question arises: What is the relation between the theories of controllers for stochastic systems in robust control and H_∞ control theory? A detailed answer to this question has been provided in Section 7 of the paper. As it turned out, in H_∞ control theory, of interest are the operator $H_{zv} : v \mapsto z$ and its induced norm; in robust control theory, the operator $\Delta : z \mapsto v$ and its norm $\|\Delta\|$, which plays a crucial role when determining the stability radius of an uncertain system. Various aspects of the linear theory of stochastic robust systems were reflected in the monograph [37].

The theory of multiplicative stochastic systems presented in the review is some generalization of the linear theory since the diffusion term in the state equation is taken here as multiplicative instead of linear. An unconventional approach to the theory of Gaussian systems has been described in Section 4. Also, note interesting results of the stochastic theory of controllers based on the observed output y, also discussed in the review. Here, we have to pass to an augmented system with the state vector $\bar{x} = (x, \hat{x})$, where $\hat{x}$ is the controller's state vector calculated from the output y. After that, it seems interesting to compare the theory of such controllers with those of deterministic ones and controllers based on the state vector x.

Finally, let us emphasize possible lines to develop further the stochastic H^2/H_∞ control theory and its generalizations. In this context, we mention works on control theory for time-invariant systems with bounded spectral characteristics [38], some classes of nonlinear systems [39], both robust [40] and nonrobust, and systems with non-Gaussian uncertainties [41] and incomplete information about the state vector [42, 43]. Research into the control theory of discrete systems continues as well.

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